ISRAEL JOURNAL OF MATHEMATICS 76 (1991), 183–192

# NOTES ON LACUNARY MÜNTZ POLYNOMIALS

BY

PETER BORWEIN

*Department of Mathematics, Dalhousie University, Halifax Nova Scotia, Canada B3H 3J5*

AND

TAMÁS ERDÉLYI

*Department of Mathematics, The Ohio State University Columbus, OH 43210, USA*

ABSTRACT

We prove that a Müntz system has Chebyshev polynomials on $[0,1]$ with uniformly bounded coefficients if and only if it is lacunary. A sharp Bernstein-type inequality for lacunary Müntz systems is established as well. As an application we show that a lacunary Müntz system fails to be dense in $C(A)$ in the uniform norm for every $A \subset [0,1]$ with positive outer Lebesgue measure. A bounded Remez-type inequality is conjectured for non-dense Müntz systems on $[0,1]$ which would solve Newman's problem concerning the density of products of Müntz systems.

## 1. Introduction and Notations

Denseness and approximation questions in Markov systems are intimately and essentially tied to the behavior of the associated Chebyshev polynomials; see, for example, [1, 2]. Our intention, in this paper, is to show that lacunary Müntz systems are completely characterized by the property that their associated Chebyshev polynomials on $[0,1]$ have uniformly bounded coefficients. This is the content of Theorems 2.1 and 2.2. This allows us to give an (essentially) sharp Bernstein-type inequality (Theorem 3.1) for these systems, and from this we can rederive a version of a Müntz-type theorem in [2] (Theorem 4.1). This theorem

Received March 3, 1991

tells us that, under the assumption of lacunarity, a Müntz system fails to be dense in $C(A)$, where $A \subset [0,\infty)$ is any set with positive Lebesgue outer measure. We conjecture that this extension of the Müntz-Szász theorem holds in any non-dense Müntz system. In Section 5 a bounded Remez-type inequality is conjectured for non-dense Müntz systems on $[0,1]$ which would solve Newman's problem concerning the density of products of Müntz systems.

Let $\Lambda = \{\lambda_i\}_{i=0}^{\infty}$, $0 \le \lambda_0 < \lambda_1 < \cdots$. The set of all Müntz polynomials of the form $p(x) = \Sigma_{j=0}^{n} a_j x^{\lambda_j}$ with real coefficients $a_j$ will be denoted by $H_n(\Lambda)$. Let $H(\Lambda) = \bigcup_{n=0}^{\infty} H_n(\Lambda)$. The $n$-th Chebyshev polynomial $T_n$ of $H(\Lambda)$ on $[0,1]$ is defined by the properties

(1) $T_n \in H_n(\Lambda)$,
(2) $T_n$ equioscillates $n+1$ times on $[0,1]$,
(3) $\max_{0\le x\le 1} |T_n(x)| = 1$,
(4) $T_n(1) = 1$.

To be precise, property (2) means that $T_n(x)$ achieves the values

$$\pm \max_{0\le x\le 1} |T_n(x)| = \pm 1$$

$n+1$ times on $[0,1]$ with alternating signs. It is well-known that such a $T_n$ $(n = 0,1,\ldots)$ exists and it is unique.

## 2. Chebyshev Polynomials of $H(\Lambda)$ on $[0,1]$ with Uniformly Bounded Coefficients

Let

$$T_n(x) = \sum_{j=0}^{n} a_{j,n} x^{\lambda_j}$$

be the $n$-th Chebyshev polynomial of $H(\Lambda)$ on $[0,1]$. We characterize the Müntz systems $H(\Lambda)$ for which $|a_{j,n}| \le K(\Lambda)$ for every $j = 0,1,\ldots,n$; $n = 0,1,\ldots$, where $K(\Lambda)$ is a constant depending only on $\Lambda$.

THEOREM 2.1: *There is a constant $c_1(\Lambda)$ depending only on $\Lambda$ such that $|a_{j,n}| \le c_1(\Lambda)$ for every $j = 0,1,\ldots,n$; $n = 0,1,\ldots$ if and only if there is a constant $c_2(\Lambda) > 1$ depending only on $\Lambda$ such that $\lambda_{i+1}/\lambda_i \ge c_2(\Lambda)$ for every $i = 1,2,\ldots$.*

In fact, in one of the directions we will prove more. Namely we have

THEOREM 2.2: *If $\lambda_{i+1}/\lambda_i \geq c_2(\Lambda) > 1$ for every $i = 1, 2, \dots$ with some constant $c_2(\Lambda)$ depending only on $\Lambda$, then there is a constant $c_1(\Lambda)$ depending only on $\Lambda$ such that*

$$|b_{j,n}| \leq c_1(\Lambda) \max_{0 \leq x \leq 1} |p(x)| \qquad (j = 0, 1, \dots, n; n = 0, 1, \dots)$$

*for every*

$$p(x) = \sum_{j=0}^{n} b_{j,n} x^{\lambda_j} \in H_n(\Lambda).$$

## 3. A Bernstein-type Inequality for Lacunary Müntz Systems

The following pretty inequalities were proved by D. Newman [9]. We have

$$\frac{2}{3} \sum_{i=0}^{n} \lambda_i \leq \sup_{p \in H_n(\Lambda)} \frac{|p'(1)|}{\max_{0 \leq x \leq 1} |p(x)|} \leq 11 \sum_{i=0}^{n} \lambda_i. \tag{3.1}$$

In the case that $\Sigma_{i=1}^{\infty} \lambda_i^{-1} < \infty$, $\lambda_0 = 0$, $\lambda_1 \geq 1$ and $\inf\{\lambda_{i+1} - \lambda_i : i \in \mathbb{N}\} > 0$, Lemma 2 of [1] gives

$$\max_{0 \leq x \leq y} |p'(x)| \leq c(\Lambda, y) \max_{0 \leq x \leq 1} |p(x)| \tag{3.2}$$

for every $p \in H(\Lambda)$ and $0 \leq y < 1$, where $c(\Lambda, y)$ is a constant depending only on $\Lambda$ and $y$. In our next theorem we prove that if $\Lambda$ is lacunary, $\lambda_0 = 0$ and $\lambda_1 \geq 1$, then in (3.2) $c(\Lambda, y)$ can be replaced by $c_3(\Lambda)/(1-y)$, where $c_3(\Lambda)$ is a constant depending only on $\Lambda$.

THEOREM 3.1: *If $\lambda_0 = 0$, $\lambda_1 \geq 1$ and $\lambda_{i+1}/\lambda_i \geq c_2(\Lambda) > 1$ for every $i = 1, 2, \dots$ with some constant $c_2(\Lambda)$ depending only on $\Lambda$, then there is a constant $c_3(\Lambda)$ depending only on $\Lambda$ such that*

$$|p'(y)| \leq \frac{c_3(\Lambda)}{1-y} \max_{0 \leq x \leq 1} |p(x)|$$

*for every $p \in H(\Lambda)$ and $0 \leq y < 1$.*

## 4. A Müntz-type Theorem for Lacunary Müntz Systems on Every $A \subset [0,1]$ with Positive Outer Lebesgue Measure

A beautiful theorem of Müntz of Szász [3, 7] states that a Müntz system $H(\Lambda)$ with $\lambda_0 = 0$ and $\lambda_i \nearrow \infty$ is dense in $C([0,1])$ in the uniform norm if and only if $\Sigma_{i=1}^{\infty} \lambda_i^{-1} = \infty$. In 1943 Clarkson and Erdös [4] showed that if $\inf\{\lambda_{i+1} - \lambda_i : i \in \mathbb{N}\} > 0$, then for an arbitrary $[a,b] \subset (0,\infty)$ a Müntz system $H(\Lambda)$ with $\lambda_i \nearrow \infty$ is dense in $C([a,b])$ in the uniform norm if and only if $\Sigma_{i=1}^{\infty} \lambda_i^{-1} = \infty$. The following conjecture seems to be hard.

CONJECTURE 4.1: *Suppose that $A \subset [0,1]$ is a closed set with positive Lebesgue measure. Then a Müntz system $H(\Lambda)$ with $\lambda_0 = 0$ and $\lambda_i \nearrow \infty$ is dense in $C(A)$ in the uniform norm if and only if $\Sigma_{i=1}^{\infty} \lambda_i^{-1} = \infty$.*

We prove an application of Theorem 3.1 for lacunary Müntz systems.

THEOREM 4.2: *If $\lambda_0 = 0$ and $\lambda_{i+1}/\lambda_i \geq c_2(\Lambda) > 1$ for every $i = 1, 2, \ldots$ with some constant $c_2(\Lambda)$ depending only on $\Lambda$, then $H(\Lambda)$ fails to be dense in $C(A)$ in the uniform norm for any $A \subset [0,1]$ with positive Lebesgue outer measure.*

## 5. Remez-type Inequalities for Müntz Systems

In [2] we pointed out that Conjecture 4.1 would trivially follow from the following Remez-type inequality.

CONJECTURE 5.1: *Let $\Sigma_{i=1}^{\infty} \lambda_i^{-1} < \infty$. For every $0 < s < 1$ there is a constant $c(s,\Lambda)$ depending only on $s$ and $\Lambda$ such that $|p(0)| \leq c(s,\Lambda)$ for every $p \in H(\Lambda)$ with $m(\{x \in [0,1] : |p(x)| \leq 1\}) \geq s$, where $m(\cdot)$ denotes the Lebesgue measure.*

By the already mentioned Müntz-Szász theorem, such a bounded Remez-type inequality cannot hold when $\Sigma_{i=1}^{\infty} \lambda_i^{-1} = \infty$. A discussion of Remez-type inequalities for algebraic and trigonometric polynomials can be found in [5] and [6]. In [2] we proved Conjecture 5.1 in the case when $\lambda_j > \lambda^j$, $j = 1, 2, \ldots$ with some $\lambda > 1$, and obtained Theorem 4.2 as a consequence of it. Our proof of Theorem 4.2 in Section 6 will be essentially shorter and it may open other ways to attack Conjecture 4.1.

There was another motivation to establish Conjecture 5.1. It would solve Newman's problem [10, P(10.5), p. 50] concerning the density of the classes

$$H^k(\Lambda) = \left\{ p = \prod_{j=1}^{k} p_j : \; p_j \in H(\Lambda), \; j = 1, 2, \ldots, k \right\}$$

in $C([0,1])$ in the uniform norm, when $\lambda_j = j^2$, $j = 1, 2, \ldots$. Namely, if Conjecture 5.1 were true, then $H^k(\Lambda)$ would fail to be dense in $C([0,1])$ in the uniform norm for every $k \in \mathbb{N}$, whenever $\Sigma_{i=1}^{\infty} \lambda_i^{-1} < \infty$. Indeed, Conjecture 5.1 implies that

$$(5.1) \qquad m(\{x \in [0,1] : |q(x)| \geq \alpha^{-1}|q(0)|\}) \geq 1 - (2k)^{-1}$$

for every $q \in H(\Lambda)$ with $\alpha = c((2k)^{-1}, \Lambda) + 1$. Hence

$$(5.2) \qquad m(\{x \in [0,1] : p(x)| \geq \alpha^{-k}|p(0)|\}) \geq \frac{1}{2}$$

for every $p \in H^k(\Lambda)$ (if $p = p_1 p_2 \cdots p_k$ with $p_j \in H(\Lambda)$, $j = 1, 2, \ldots, k$, then $|p(x)| \geq \alpha^{-k}|p(0)|$ holds for every $x \in [0,1]$ satisfying $|p_j(x)| \geq \alpha^{-1}|p_j(0)|$ for each $j = 1, 2, \ldots, k$). Now let $f \in C([0,1])$ be such that $f(x) = 0$ if $1/4 \leq x \leq 1$, and $f(0) = 1$. If there were a $p \in H^k(\Lambda)$ such that

$$(5.3) \qquad \max_{0 \leq x \leq 1} |p(x) - f(x)| \leq \frac{1}{2}\alpha^{-k},$$

then it would contradict (5.2). Similarly, Conjecture 5.1 would imply that if $\Sigma_{i=1}^{\infty} \lambda_i^{-1} < \infty$, and $A \subset [0,1]$ is of positive measure, then $H^k(\Lambda)$ is not dense in $C(A)$ for every $k \in \mathbb{N}$.

## 6. Proofs

To prove Theorem 2.2 we need the following result of Hardy and Littlewood [8].

THEOREM A: *Assume that $\gamma_0 = 0$, $\gamma_{i+1}/\gamma_i \geq \eta > 1$ for every $i = 1, 2, \ldots$, $f(x) = \Sigma_{i=0}^{\infty} b_i x^{\gamma_i}$ is convergent in $[0,1)$ and $\lim_{x \to 1-} f(x) = A$ exists. Then $A = \Sigma_{i=0}^{\infty} b_i$.*

*Proof of Theorem 2.2:* Without loss of generality we may assume that $\lambda_0 = 0$. Assume indirectly that there are

$$(6.1) \qquad P_k(x) = \sum_{j=0}^{n_k} b_{j,n_k} x^{\lambda_j}$$

such that

$$(6.2) \qquad \max_{0 \leq x \leq 1} |P_k(x)| = 1, \quad k = 1, 2 \ldots$$

and

$$\max_{0\le j\le n_k} |b_{j,n_k}| \ge k^4, \quad k=1,2,\dots. \tag{6.3}$$

Choose a sequence $\{\alpha_k\}_{k=1}^{\infty}$ of positive integers such that

$$\alpha_1 = 1 \quad \text{and} \quad \alpha_{k+1} \ge 2\alpha_k \lambda_{n_k} \quad \text{for } k=1,2,\dots. \tag{6.4}$$

Now let

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} P_k(x^{\alpha_k}). \tag{6.5}$$

Note that the above sum converges uniformly on $[0,1]$ because of (6.2). Therefore $f$ is continuous on $[0,1]$. For the sake of brevity let

$$N_0 = 0 \quad \text{and} \quad N_k = \sum_{i=1}^{k} n_i, \quad k=1,2,\dots. \tag{6.6}$$

Further let

$$\gamma_0 = 0, \qquad b_0 = \sum_{k=1}^{\infty} \frac{1}{k^2} b_{0,n_k} \tag{6.7}$$

and

$$\gamma_{N_{k-1}+j} = \alpha_k \lambda_j, \quad j=1,2,\dots,n_k \quad \text{and} \quad k=1,2,\dots \tag{6.8}$$

Observe that the sum in (6.7) converges, since by (6.2) $|b_{0,n_k}| = |P_k(0)| \le 1$. Also, from $\lambda_{i+1}/\lambda_i \ge c_2(\Lambda) > 1$, $i=1,2,\dots$, and (6.4) we can deduce that $\gamma_{i+1}/\gamma_i \ge \eta > 1$, $i=1,2,\dots$ with $\eta = \min\{c_2(\Lambda), 2\}$. Let $\Lambda' = \{\gamma_i\}_{i=1}^{\infty}$. Then by (6.5) $f \in C([0,1])$ can be approximated by Müntz polynomials from $H(\Lambda')$ with arbitrary accuracy. Hence by Theorem 3 of [4] $f$ is of the form

$$f(x) = \sum_{i=0}^{\infty} b_i x^{\gamma_i}, \tag{6.9}$$

where the sum converges in $[0,1)$. Since $f$ is continuous on $[0,1]$, Theorem A implies that

$$\sum_{i=0}^{\infty} b_i = A \tag{6.10}$$

exists. By Theorem 3 of [4] each $b_{j,n_k}$ ($j = 1,2,\ldots,n_k$; $k = 1,2,\ldots$) is equal to one of the coefficients $b_i$ ($i = 1,2,\ldots$). Since $|b_{0,n_k}| = |P_k(0)| \le 1$ for every $k = 1,2,\ldots$, from (6.3) and (6.5) we deduce that for every $k \in \mathbb{N}$ there is an $i \in \mathbb{N}$ such that $|b_i| \ge k^2$. This contradicts (6.10), thus the theorem is proved. ∎

*Proof of Theorem 2.1:* We show that if $\liminf_{i\to\infty} \lambda_i/\lambda_{i-1} = 1$, then there is no $c_1(\Lambda)$ such that $|a_{j,n}| \le c_1(\Lambda)$ for every $j = 0,1,\ldots,n$; $n = 0,1,\ldots$. To see this, for an arbitrary $\varepsilon > 0$ we select an $n \in \mathbb{N}$ such that $\lambda_{n-1}/\lambda_n > 1 - \varepsilon$. Observe that $P_n(x) = x^{\lambda_n} - x^{\lambda_{n-1}}$ achieves its maximum modulus on $[0,1]$ at

$$x = \left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{1/(\lambda_n - \lambda_{n-1})},$$

hence

$$\max_{0\le x\le 1} |P_n(x)| \le \left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{\frac{\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}} \left(1 - \frac{\lambda_{n-1}}{\lambda_n}\right) \le 1 - \frac{\lambda_{n-1}}{\lambda_n} < \varepsilon,$$

which shows that the leading coefficient of the $n$-th Chebyshev polynomial $T_n$ of $H(\Lambda)$ on $[0,1]$ is at least $1/\varepsilon$, otherwise

$$\frac{1}{a_{n,n}} T_n - P_n \in H_{n-1}(\Lambda)$$

would have at least $n$ zeros in $(0,1)$, a contradiction. ∎

*Proof of Theorem 3.1:* Let $p = H(\Lambda)$ be of the form

$$p(x) = b_{0,n} + \sum_{j=1}^{n} b_{j,n} x^{\lambda_j} \tag{6.11)}$$

such that

$$\max_{0\le x\le 1} |p(x)| \le 1. \tag{6.12}$$

From $\lambda_{i+1}/\lambda_i \ge c_2(\Lambda) > 1$ ($i = 1,2,\ldots$), $\lambda_1 \ge 1$, Theorem 2.2 and (6.12), we obtain

$$\begin{aligned} |p'(y)| &= \Big|\sum_{j=1}^{n} b_{j,n}\lambda_j y^{\lambda_j-1}\Big| \le \sum_{j=1}^{n} |b_{j,n}|\lambda_j y^{\lambda_j-1} \\ &\le c_1(\Lambda)\sum_{j=1}^{n} \lambda_j y^{\lambda_j-1} \le c_3(\Lambda)\sum_{j=0}^{\infty} y^j = \frac{c_3(\Lambda)}{1-y}, \end{aligned} \tag{6.13}$$

which yields the theorem. ∎

To prove Theorem 4.1 we need two lemmas.

LEMMA 6.1: *If $\lambda_0 = 0$ and $\lambda_{i+1}/\lambda_i \ge c_2(\Lambda) > 1$ for every $i = 1, 2, \ldots$, then for every $0 < y < 1$ there is an integer $k(y, \Lambda) > 0$ depending only on $y$ and $\Lambda$ such that the $n$-th Chebyshev polynomial $T_n$ of $H(\Lambda)$ on $[0,1]$ has at most $k(y,\Lambda)$ zeros in $[0,y]$ $(n = 1, 2, \ldots)$.*

The proof of Lemma 6.1 can be found in [1, Theorem 3].

LEMMA 6.2: *Let $\lambda_0 = 0$, $\lambda_1 \ge 1$ and $\lambda_{i+1}/\lambda_i \ge c_2(\Lambda) > 1$ for every $i = 1, 2, \ldots$. Denote the extreme points of the $n$-th Chebyshev polynomial $T_n$ of $H(\Lambda)$ on $[0,1]$ by $1 = y_{0,n} > y_{1,n} > \cdots > y_{n,n} = 0$. Then there is a constant $c = c_4(\Lambda) > 0$ depending only on $\Lambda$ such that*

$$(6.14)\quad T_n(x) \ge \frac{1}{2} \quad \text{if} \quad y_{j,n} \le x \le y_{j,n} + c(1 - y_{j,n}), \qquad 1 \le j \le n,\ j \text{ is even}$$

*and*

$$(6.15)\quad T_n(x) \le -\frac{1}{2} \quad \text{if} \quad y_{j,n} \le x \le y_{j,n} + c(1 - y_{j,n}), \qquad 1 \le j \le n,\ j \text{ is odd.}$$

*Proof of Lemma 6.2:* The proof is a straightforward combination of the equioscillation of the Chebyshev polynomials $T_n$, the Mean Value Theorem and Theorem 3.1. ∎

*Proof of Theorem 4.2:* Without loss of generality we may assume that $\lambda_1 \ge 1$; the case $\lambda_1 > 0$ can be obtained from this by the scaling $x \to x^{1/\lambda_1}$. Denote the Lebesgue outer measure of a set $A \subset [0,1]$ by $m(A)$. If $m(A) > 0$ and $A \subset [0,1]$, then by the Lebesgue Density Theorem there is an $0 < a \in A$ such that the left hand side density of $A$ at $a$ is 1. By the scaling $x \to x/a$, without loss of generality we may assume that $a = 1$, that is

$$(6.16)\qquad \lim_{x \to 0+} \frac{m((1-x, 1) \cap A)}{x} = 1.$$

Hence, there is a $0 < y < 1$ such that

$$(6.17)\qquad \frac{m((1-x) \cap A)}{x} \ge \max\{1 - c/2, 3/4\} \quad \text{for every } 0 < x \le y,$$

where $c = c_4(\Lambda) > 0$ is the same as in Lemma 6.2. By Lemma 6.1 there is an integer $k = k(y/2, \Lambda) > 0$ such that for the extreme points $1 = y_{0,n} > y_{1,n} > \cdots > y_{n,n} = 0$ of the $n$-th Chebyshev polynomial $T_n$ of $H(\Lambda)$ on $[0,1]$ we have

$$(6.18)\qquad y_{j,n} > 1 - y/2 \quad \text{if} \quad 0 \le j \le n - k.$$

Since $m((1-y, 1-y/2) \cap A) > 0$ (see (6.17)), there are $k+3$ distinct points $a_1 < a_2 < \cdots < a_{k+3}$ in $(1-y, 1-y/2) \cap A$. Now let $g$ be a continuous function on $[0,1]$ (and hence on $A$) such that

$$g(x) = 0 \quad \text{if} \quad 1 - y/2 \le x \le 1 \tag{6.19}$$

and

$$g(a_j) = 2(-1)^j \quad \text{for every } j = 1, 2, \ldots, k+3. \tag{6.20}$$

Assume that there is a $p \in H(\Lambda)$ such that

$$\max_{x \in A} |p(x) - g(x)| \le \tfrac{1}{4}. \tag{6.21}$$

We will show that $p - T_n \in H_n(\Lambda)$ has at least $n+1$ different zeros in $(0,1)$, a contradiction. Indeed, it follows from Lemma 6.2 and (6.17) that there are $n-k$ distinct points $z_{1,n} > z_{2,n} > \cdots > z_{n-k,n}$ from $A$ such that

$$T(z_{j,n}) \ge \tfrac{1}{2} \quad \text{if} \quad 1 \le j \le n-k \text{ and } j \text{ is even} \tag{6.22}$$

and

$$T(z_{j,n}) \le -\tfrac{1}{2} \quad \text{if} \quad 1 \le j \le n-k \text{ and } j \text{ is odd.} \tag{6.23}$$

Now (6.19) – (6.23) and $\max_{0 \le x \le 1} |T_n(x)| = 1$ imply

$$(p - T_n)(z_{j,n}) < 0 \quad \text{if} \quad 1 \le j \le n-k \text{ and } j \text{ is even,} \tag{6.24}$$

$$(p - T_n)(z_{j,n}) > 0 \quad \text{if} \quad 1 \le j \le n-k \text{ and } j \text{ is odd,} \tag{6.25}$$

$$(p - T_n)(a_j) < 0 \quad \text{if} \quad 1 \le j \le k+3 \text{ and } j \text{ is odd,} \tag{6.26}$$

and

$$(p - T_n)(z_{j,n}) > 0 \quad \text{if} \quad 1 \le j \le k+3 \text{ and } j \text{ is even.} \tag{6.27}$$

From (6.24) – (6.27) we can deduce that $p - T_n$ has at least $k+2$ zeros in $(a_1, a_{k+3}) \subset (1-y, 1-y/2)$. Thus $p - T_n \in H_n$ has at least $n+1$ different zeros in $(0,1)$, but $p \not\equiv T_n$, since $\max_{0 \le x \le 1} |p(x)| \ge 7/4$ and $\max_{0 \le x \le 1} |T_n(x)| = 1$. This is a contradiction which finishes the proof. ∎

## References


1. P. Borwein, *Zeros of Chebyshev polynomials in Markov systems*, J. Approx. Theory **63** (1990), 56–64.
2. P. Borwein and T. Erdélyi, *Lacunary Müntz systems*, manuscript.
3. E. W. Cheney, *Introduction to Approximation Theory*, McGraw-Hill, New York, 1966.
4. J. A. Clarkson and P. Erdös, *Approximation by polynomials*, Duke Math. J. **10** (1943), 5–11.
5. T. Erdélyi, *The Remez inequality on the size of polynomials*, in *Approximation Theory VI* **1**, (C. K. Chui, L. L. Schumaker and J. D. Ward, eds.), Academic Press, Boston, 1989, pp. 243–246.
6. T. Erdélyi, *Remez-type inequalities on the size of generalized polynomials*, J. London Math. Soc., to appear.
7. R. P. Feinerman and D. J. Newman, *Polynomial Approximation*, Williams and Wilkins, Baltimore, Md., 1976.
8. G. H. Hardy and J. E. Littlewood, *A further note on the converse of Abel's theorem*, Proc. London Math. Soc. **25** (1926), 219–236.
9. D. J. Newman, *Derivative bounds for Müntz polynomials*, J. Approx. Theory **18** (1976), 360-362.
10. D. J. Newman, *Approximation with Rational Functions*, Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics, No. 41, Amer. Math. Soc., Providence, Rhode Island, 1979.